Multiple Traces Boundary Integral Formulation for Helmholtz Transmission Problems

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We present a boundary formulation of the Helmholtz transmission problem over multiple penetrable subdomains that lends itself to operator preconditioning via Calderón projectors. Composite scatterers for scalar elliptic operators are first tackled in [3] but the proposed first kind integral formulation is not well-suited for preconditioning. An alternative is given in [2] based on the tearing and interconnecting technique developed in the context of non-overlapping domain decomposition methods. Although it can readily be preconditioned, the method shows spurious modes and requires the iterative construction of Steklov-Poincaré operators as well as local and global preconditioners [1].

The presented approach relies on the weak enforcement of jump conditions across interfaces by doubling the number of trace unknowns in suitable functional spaces. Let $\partial\Omega_i$, $i = 0, \ldots, N$, denote each subdomain boundary. If $\mathbf{V}_i := H^{1/2}(\partial\Omega_i) \times H^{-1/2}(\partial\Omega_i)$, our formulation is set on subspaces $\widetilde{\mathbf{V}}_i \subset \mathbf{V}_i$, for which restriction and extension by zero operations are well defined. Through the use of interior Calderón projectors, the problem is cast in variational Galerkin form with a matrix operator whose diagonal is composed of block boundary integral operators. Specifically, let $\mathbb{V}_N := \mathbf{V}_0 \times \cdots \times \mathbf{V}_N$ and equivalently for $\widetilde{\mathbb{V}}_N$. We seek $\boldsymbol{\lambda} \in \widetilde{\mathbb{V}}_N$ such that the variational form:

$$(\mathsf{M}_N\, \boldsymbol{\lambda}\,,\, \boldsymbol{\varphi})_\times = \frac{1}{2}\left(\begin{pmatrix} \mathsf{X}_0\, \mathbf{g} \\ -\,\mathsf{R}_{10}^\dagger\, \mathsf{R}_{01}\, \mathbf{g} \\ \vdots \\ -\,\mathsf{R}_{N0}^\dagger\, \mathsf{R}_{0N}\, \mathbf{g} \end{pmatrix}, \boldsymbol{\varphi}\right)_\times \qquad \text{for all } \boldsymbol{\varphi} \in \widetilde{\mathbb{V}}_N \tag{1}$$

is satisfied for $\mathbf{g} \in \widetilde{\mathbf{V}}_0$, Dirichlet and Neumann data on the exterior boundary, with R_{0N} and $\mathsf{R}_{0N}^\dagger$ being restriction and extension by zero operators, and

$$\mathsf{M}_N := \begin{pmatrix} \mathsf{A}_0 & -\frac{1}{2}\widetilde{\mathsf{X}}_{01} & \cdots & -\frac{1}{2}\widetilde{\mathsf{X}}_{0N} \\ -\frac{1}{2}\widetilde{\mathsf{X}}_{10} & \mathsf{A}_1 & \cdots & -\frac{1}{2}\widetilde{\mathsf{X}}_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{2}\widetilde{\mathsf{X}}_{N0} & -\frac{1}{2}\widetilde{\mathsf{X}}_{N1} & \cdots & \mathsf{A}_N \end{pmatrix} : \widetilde{\mathbb{V}}_N \longrightarrow \mathbb{V}_N. \tag{2}$$

where operators $\widetilde{\mathsf{X}}_{ij}$ account for transmission at the common interfaces.

We show uniqueness of solutions, continuity and coercivity in a larger space. Finally, numerical results validate the model and its amenability to different kinds of preconditioning.

References

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