Stabilized Galerkin Methods for General Advection-Diffusion Problems

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Our starting point is the observation that the linear 2nd-order convection-diffusion problem

$$-\epsilon \mathsf{div}\mathbf{grad} u + \boldsymbol{\beta} \cdot \mathbf{grad} u = f \quad , \quad \text{in } \Omega \subset \mathbb{R}^n \tag{1}$$

for an unknown scalar function u is one instance of the class of general advection diffusion problems

$$-\epsilon(-1)^l d * d\omega + *L_{\boldsymbol{\beta}}\omega = \phi \tag{2}$$

for unknown l-forms ω. The symbol $*$ stands for a Hodge operator mapping a l-form to a $n-l$-form, and d denotes the exterior derivative. By $L_{\boldsymbol{\beta}}$ we denote the Lie derivative of ω for the given velocity field $\boldsymbol{\beta}$ [?]. We want to stress that besides the case $l = 0$ the cases $l = 1$ and $l = 2$ in $\mathbb{R}^3$ a relevant for numerical modeling, e.g. in MHD [?] or magnetic convection-diffusion [?]. For $0 < \varepsilon \ll 1$ we encounter in (??) singularly perturbed boundary value problems, whose discretization have to be treated carefully. While the stable discretization of (??) has attracted immense attention in numerical analysis, there is little research for the other cases.

We present a uniform framework of Galerkin formulations for the general advection-diffusion problem (??) for approximation spaces which are either conforming or non-conforming. Except for $l = 0$ and conforming approximation spaces these formulations can be seen as a kind of upwind discretization, featuring upwind numerical flux functions. Hence they are stabilized formulations according to the terminology used for discontinuous Galerkin methods. Numerical experiments give evidence that for the case $l > 0$ we obtain the desirable properties of stabilized schemes: even if the analytic solution exhibits steep boundary layers the numerical solution does not suffer from a pollution effect. This is due to a non-standard treatment of the Dirichlet boundary conditions.

References

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