GETTING EVEN WITH THE MAXWELL TENSOR, REDUX

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1. Overview

This is part of a series of studies of the question of forces in electromagnetism, where I try to recast the theory in modern differential geometric formalism, with the objective of identifying suitable "objects", in the sense of Object Oriented Programming, and to find discrete representations, computer-adapted, of these objects. A typical example is the formulation of magnetostatics in terms of a 2-form b and a twisted 1-form h, whose discrete representations are (real- or complex-valued) arrays $\mathbf{b}$ and $\mathbf{h}$, indexed over faces of a finite-element mesh, with Whitney forms providing the necessary link between $\{b, h\}$ and $\{\mathbf{b}, \mathbf{h}\}$, as well as a mechanism to build convergence proofs and error analysis.

Extending this working programme to the question of forces raises many challenges, if only because the question (here very vaguely phrased) of "what is the force exerted by the magnetic field over a body" cannot be properly stated in the restricted framework of Maxwell's theory: It requires a theory of the *coupled* problem, electromagnetism plus elasticity, which forces us to examine the geometric status of objects such as strain and stress, and to discretize them in a manner that be compatible with what is done in the "electromagnetic compartment" of the coupled system. It also requires a knowledge of coupled constitutive laws, which is primarily a measurement problem, but the question of how to express the benchmark-observed constitutive laws in differential-geometric language is still outstanding.

At some stage in this endeavor, one has to deal with the Maxwell tensor. This object *has* an expression in terms of the differential forms h and b, and does *not* depend on state variables that belong to the mechanical compartment of the coupled system. Yet, it is reputed to provide magnetic forces, and some parts of the literature strongly suggest that Maxwell's tensor is the primary object of the theory, from which the local force field (not only its integrals over finite regions) could be *derived*. How could that be, if elastic behavior is ignored?

The virtual power principle (VPP), as we shall see, clarifies these issues, and helps find the boundary of the realm over which Maxwell's tensor reigns. Shortly said, Maxwell's tensor is logically weaker than the VPP, from which it can be derived, and it fails when *magnetostrictive* effects exist. On the other hand, it gives correct answers when only *shape effects* are to be considered. The mathematical analysis will provide precise characterizations of these two aspects of magneto-elastic interaction.

2. Development

Strain ε and stress σ can be considered (with minor departures from the tradition of Mechanics) as, respectively, a vector-valued 1-form and a twisted covector-valued 2-form. Strain ε describes the difference between the actual placement and the reference placement of a given material vector. Stress σ gives the flux of momentum across a material bivector. This way, coupled equations for electrodynamics and elastodynamics are (with d for the exterior derivative)

(1) $$-\partial_t d + \mathsf{d}h = j, \quad \partial_t b + \mathsf{d}e = 0, \quad \partial_t p - \mathsf{d}\sigma = f, \quad -\partial_t \varepsilon + \mathsf{d}v = 0,$$

where p (a twisted covector-valued 3-form) is the momentum density and v (a vector field, but here considered as a vector-valued 0-form) is the material velocity. Forcing terms, describing what the outer world inputs to the coupled system, are the current density j and the applied force f. (We ignore losses due to Joule effect and friction phenomena, for the sake of brevity only.) To close (1), one needs an "energy functional" $\Psi(b, d, p, \varepsilon)$, accounting for all electrical and mechanical properties of the system's bodies, from which constitutive laws are simply derived by Fréchet differentiation:

(2) $$h = \partial_b\Psi, \quad e = \partial_d\Psi, \quad v = \partial_p\Psi, \quad \sigma = \partial_\varepsilon\Psi.$$

A simple calculation (wedge-multiply by e, h, v and $-\sigma$ in (1), add, integrate over all space) then shows that Ψ *deserves* to be called "energy". Restricting to elasto-magneto-*statics*, one has

(3) $$\mathrm{d}b = 0, \quad h = \partial_b\Psi, \quad \mathrm{d}h = j, \quad \mathrm{d}\varepsilon = 0, \quad \sigma = \partial_\varepsilon\Psi, \quad -\mathrm{d}\sigma = f,$$

with now Ψ a function of b and ε only. This is a well-posed problem (for which a natural discretization exists). *No invocation of any "magnetic force" appears necessary.*

To make this force emerge from the formalism, one may use the natural splitting $\Psi(b, \varepsilon) = \Psi_{ela}(\varepsilon) + \Psi_{mag}(b, \varepsilon)$ of energy into two parts, where "elastic" energy $\Psi_{ela}(\varepsilon)$ is defined as $\Psi(0, \varepsilon)$, and "magnetic" energy as $\Psi(b, \varepsilon) - \Psi(0, \varepsilon)$. Setting them $\sigma_{ela} = \partial_\varepsilon\Psi_{ela}(\varepsilon)$, one transforms the last eq. (3) into $-\mathrm{d}\sigma_{ela} = f + \mathrm{d}(\partial_\varepsilon\Psi_{mag}(b, \varepsilon))$. The last term, a covector-valued twisted 3-form, can then be interpreted as encoding the momentum transfer from the "magnetic compartment" of the coupled system to its "elastic compartment", that is to say, as the density of force of magnetic origin exerted on matter. So it all goes as if magnetic force was added to applied force f. This approach leads to the classic expressions of this force density [1], that is to say, in terms of proxies, $J \times B$, as well as $^1/_2\, B \cdot (\nabla\nu) \cdot B$ when the ν in $B = \nu H$ depends on position, or else $^1/_2\, B \cdot (\partial_\varepsilon\nu) \cdot B$ when ν depends on the *local* value of the strain. (Reluctivity $\nu(x)$ at point x depends on the *whole* field ε to the extent that the displacement u, obtained by integrating ε in space, determines which kind of matter sits at point x, and hence, which value of ν applies there. This is "shape effect". In addition, $\nu(x)$ may also depend on $\varepsilon(x)$, locally: this remark can be used to characterize magnetostriction.)

Note that differentiation of $\Psi_{mag}(b, \varepsilon)$ with respect to ε amounts to study the limit at $t = 0$ of $\Psi_{mag}(b, \varepsilon + t\,\mathrm{d}v) - \Psi_{mag}(b, \varepsilon)$, where v is a virtual displacement. So what we just described is actually the VPP, which thus solves the problem of forces, provided that Ψ can accurately be determined. (In most textbook cases, one can do that from basic principles. But the interesting cases will often require an experimental determination.)

The VPP leads also to the concept of Maxwell stress: For that, consider a domain D enclosed by a surface S, and build a virtual displacement v equal to some vector V for all points in D (the same V all over D), to 0 outside D, except for a transition layer in which v goes smoothly from 0 to V. Then take the limit of the virtual work when the width of this layer tends to zero: The result, it can be shown, is $V \cdot F(D)$, where $F(D)$ is the total force over D, as given by integration over S of the standard Maxwell tensor $(B \otimes H - {}^1/_2\, B \cdot H)$. This derivation establishes the subordinate status of this object (whose expression in terms of differential forms can be found in [2]). It also helps understand why, as we shall explain, magnetostrictive effects stay beyond the reach of Maxwell's tensor: When using a *piecewise constant* virtual displacement field (except for the transition near S), one also considers a *null* virtual deformation ε, which forbids to capture variations of the energy due to local changes of ε.

References

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